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The effects of blowing and suction on free convection boundary layers on vertical surfaces with prescribed heat flux

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Abstract. The effects that blowing and suction have on the free convection boundary layer on a vertical surface with a given surface heat flux are considered. Similarity equations are derived first, their solution being dependent on the wall flux exponent n and a dimensionless transpiration parameter γ, (as well as on the Prandtl number). The range of existence of solutions is considered, with it being shown that solutions exist only for $n > -1$ for blowing, whereas they exist for all $n > n_0$ for suction, where $n_0 < -1$ and depends on γ. The solutions for strong suction and blowing are derived. In the latter case the asymptotic structure is found to be different for n in the three ranges $-1 < n < -\frac{1}{4}$, $-\frac{1}{4} < n < \frac{7}{2}$, $n > \frac{7}{2}$. Results are then obtained for the non-similarity problem of constant heat flux with a constant transpiration velocity. Solutions valid for large distances from the leading edge for both suction and blowing are derived.

1. Introduction

One method of controlling convective boundary-layer flows is to inject or to withdraw fluid through the porous bounding heated wall. This can lead to enhanced heating (or cooling) of the system and can help to delay the transition from laminar flow. Previous work on the effects of blowing and suction on free convection boundary layers has been confined to cases where there is a prescribed wall temperature. Eichhorn [1] obtained the power-law variations in wall temperature and transpiration velocity which gives rise to a similarity solution for the flow on a vertical surface. He presented results for a range of values of his (non-dimensional) transpiration parameter.

The case of uniform suction and blowing through an isothermal vertical wall was treated first by Sparrow and Cess [2]. They obtained a solution in series valid near the leading edge. This problem was considered in more detail by Merkin [3], who obtained asymptotic solutions, valid at large distances from the leading edge, for both suction and blowing. The next order corrections to the boundary-layer solution for this problem were obtained by Clarke [4], who did not invoke the usual Boussinesq approximation. The solutions for strong suction and blowing on general body shapes which admit a similarity solution has been given by Merkin [5]. A transformation of the equations for general blowing and wall temperature variations has been given by Vedhanayagam et al. [6]. The case of a heated isothermal horizontal surface with transpiration has been discussed in some detail first by Clarke and Riley [7, 8], and more recently by Lin and Yu [9].

The corresponding case of the free convection boundary-layer flow with suction or blowing through a porous wall with a prescribed heat flux has not been treated previously, and this is what we consider here. We discuss two aspects of this general problem. First, we obtain those power-law dependencies of transpiration velocity and wall heat flux which allow the governing equations to be reduced to similarity form. These ordinary differential equations are then treated in some detail. We find that their solution depends on the two basic parameters n (the exponent for the variation of wall heat flux) and γ (the non-dimensional transpiration parameter; $\gamma > 0$ for blowing, $\gamma < 0$ for suction) as well as on the Prandtl number σ. We start by considering the range of existence of solution, showing that, for blowing, a solution exists only for $n > -1$ (independent of γ and σ), whereas for suction a solution exists for $n > n_0$, where $n_0 < -1$ and where n_0 depends on γ and σ. We then go on to obtain asymptotic solutions valid for large $|\gamma|$. For suction the form of this solution is found to be independent of n. However, for blowing, the structure of this asymptotic solution depends on n, with it having an essentially different character in the three ranges $-1 < n < -\frac{1}{4}$, $-\frac{1}{4} < n < \frac{7}{2}$, $n > \frac{7}{2}$. There is an unusual and, perhaps, unexpected aspect of the solution for n is the range $-1 < n < \frac{1}{4}$ in that the boundary-layer becomes thin, with a thickness of $O(\gamma^{-1})$, for γ large. In all the other cases reported the effect of blowing is to increase the boundary-layer thickness.

In the final part of the paper we consider the, perhaps, more realistic case of uniform heat flux with uniform transpiration velocities. Here the governing boundary-layer equations cannot be reduced to similarity form and the full partial differential equations have to be solved. To do this we obtain numerical solutions starting at the leading edge and asymptotic solutions valid for large distances from the leading edge, being guided in this latter aspect by the insights gained previously from the similarity solutions.

2. Equations

The equations governing the steady free convection boundary-layer flow on a vertical surface in a fluid of kinematic viscosity ν are, on making the usual Boussinesq approximation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{1a}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta(T - T_0) + \nu\frac{\partial^2 u}{\partial y^2}\,, \tag{1b}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\nu}{\sigma}\frac{\partial^2 T}{\partial y^2}\,, \tag{1c}$$

where g is the acceleration due to gravity, β the coefficient of thermal expansion and σ the Prandtl number. Co-ordinates x and y measure distance along the wall and normal to it, with velocity components u and v respectively. T is the temperature of the fluid in the boundary layer and T_0 is the (constant) ambient temperature.

The boundary conditions to be applied are

$$v = v_w(x)\,, \quad u = 0\,, \quad k\frac{\partial T}{\partial y} = -q(x) \quad \text{on} \quad y = 0\,, \tag{2a}$$

$$u \to 0\,, \quad T \to T_0 \quad \text{as} \quad y \to \infty\,; \tag{2b}$$

k is the thermal conductivity and $v_w(x)$ and $q(x)$ are the transpiration velocity ($v_w > 0$ for blowing, $v_w < 0$ for suction) and heat flux, which can depend on the distance x from the leading edge.

To make equations (1, 2) dimensionless, we assume that $v_w(x)$ and $q(x)$ are characterized by scales v_0 and q_0 respectively. We then introduce scaled variables

$$u = U_0\bar{u}\,, \quad v = \left(\frac{\nu U_0}{l}\right)^{1/2}\bar{v}\,, \quad x = l\bar{x}\,, \quad y = \left(\frac{\nu l}{U_0}\right)^{1/2}\bar{y}\,, \quad T - T_0 = \frac{U_0^2}{g\beta l}\theta\,, \tag{3a}$$

where l is a streamwise length scale and where U_0 is a scale for the free convective flow, and is given by

$$U_0 = \left(\left(\frac{g\beta q_0}{k}\right)^2 l^3 \nu\right)^{1/5}. \tag{3b}$$

Using (3), equations (1, 2) become in dimensionless form (on dropping the bars for convenience),

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{4a}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \theta + \frac{\partial^2 u}{\partial y^2}\,, \tag{4b}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{\sigma}\frac{\partial^2 \theta}{\partial y^2}\,, \tag{4c}$$

subject to the boundary conditions

$$v = \gamma\bar{v}_w(x)\,, \quad u = 0\,, \quad \frac{\partial \theta}{\partial y} = -\bar{q}(x) \quad \text{on} \quad y = 0\,, \tag{5a}$$

$$u \to 0\,, \quad \theta \to 0 \text{ as } y \to \infty\,, \tag{5b}$$

where the dimensionless transpiration constant

$$\gamma = v_0\left(\frac{l}{\nu U_0}\right)^{1/2} = v_0\left(\frac{l^4}{\nu^2}\left(\frac{g\beta q_0}{k}\right)\right)^{1/5},$$

and where $\bar{v}_w(x)$ and $\bar{q}(x)$ are dimensionless functions of x.

Equations (4, 5) can be reduced to similarity form in the particular case when

$$\bar{v}_w(x) = x^{(n-1)/5}\,, \quad \bar{q}(x) = x^n\,, \tag{6}$$

for a given exponent n. To do this we put

$$\psi = \left(\frac{5}{n+4}\right)^{4/5} x^{(n+4)/5} f(\eta)\,, \quad \theta = \left(\frac{5}{n+4}\right)^{1/5} x^{(4n+1)/5} h(\eta)\,,$$
$$\eta = \left(\frac{n+4}{5}\right)^{1/5} y x^{(n-1)/5}\,, \tag{7}$$

where ψ is the stream function defined in the usual way. Note that we must have $n > -4$ to have forward boundary layers, in the sense discussed by Kuiken [10, 11], and hence have exponential decay in f' and θ as $\eta \to \infty$. We will now assume that $n > -4$ throughout.

Using (7), equations (4, 5) become

$$f''' + h + f'' - \left(\frac{2n+3}{n+4}\right) f'^2 = 0 \,, \tag{8a}$$

$$\frac{1}{\sigma} h'' + fh' - \left(\frac{4n+1}{n+4}\right) f'h = 0 \,. \tag{8b}$$

The boundary conditions are

$$f(0) = -\gamma \,, \quad f'(0) = 0 \,, \quad h'(0) = -1 \,, \tag{8c}$$

$$f' \to 0 \,, \quad h \to 0 \quad \text{as} \quad \eta \to \infty \,, \tag{8d}$$

where primes denote differentiation with respect to η and where γ has been rescaled by a factor $(5/(n+4))^{4/5}$. In (8c) $\gamma > 0$ for blowing and $\gamma < 0$ for suction.

We start by discussing the similarity equations (8).

3. Similarity equations

In this section we discuss the solution of equations (8a, b) subject to boundary conditions (8c, d). We assume that $n > -4$ throughout (for exponential decay of f' and θ) and start by examining the range of values of n for which a solution to equations (8) exists.

(a) Range of existence of solution

We start by integrating equation (8b) from $\eta = 0$ to $\eta \to \infty$ and applying boundary conditions (8c, d) to obtain

$$5\left(\frac{n+1}{n+4}\right) \int_0^\infty f'h \, d\eta = \frac{1}{\sigma} + \gamma h(0) \,. \tag{9}$$

(A similar result for the case $\gamma = 0$ has been given by Merkin [12].) Now, to have a realistic solution with unidirectional flow, i.e. to have $f'(\eta) \geq 0$ and $h(\eta) > 0$ on $0 \leq \eta < \infty$, the integral on the left hand side of equation (9) will be positive as will both terms on the right hand side for blowing ($\gamma > 0$). Hence, in this case, we must have

$$n > -1 \,, \quad \gamma \geq 0 \tag{10}$$

for a solution of equations (8) to exist in the required form. For suction ($\gamma < 0$) this does not apply and further considerations are required to determine the range of existence of solution.

To see that equations (8) do in fact have a solution for values of $n < -1$ we obtained a numerical solution for $\gamma = -0.5$. The results are shown in Fig. 1, where we plot $f''(0)$ and $h(0)$, related to the skin friction and wall temperature respectively, against n (for $\sigma = 1.0$). A detailed examination of the numerical results indicates that the solution terminates at a value

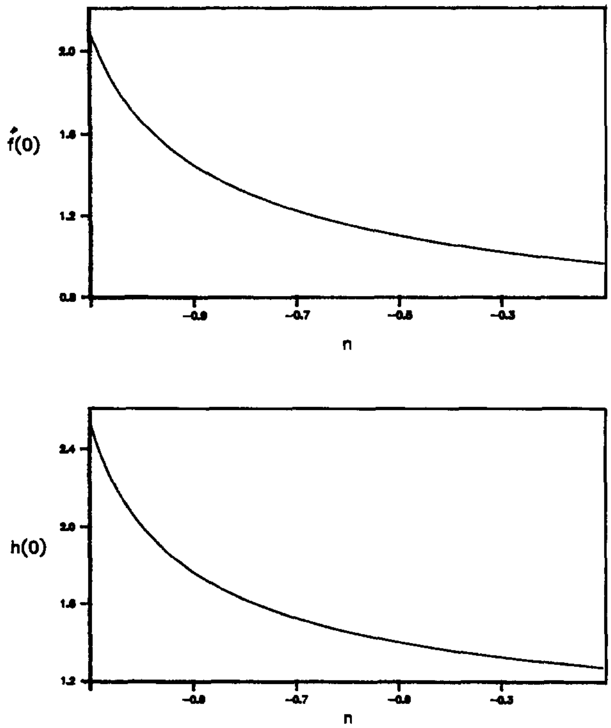

Fig. 1. Graphs of (a) $f''(0)$, (b) $h(0)$ plotted against the exponent n for suction $\gamma = -0.5$, obtained from a numerical solution of equations (8).

$n = n_0 \simeq -1.187$ and that both $f''(0)$ and $h(0)$ remain finite as $n \to n_0$, though the gradient of the curves is increasing rapidly as n_0 is approached.

To calculate the value of n_0 we proceed as follows. We put

$$n = n_0 + \varepsilon\,, \quad 0 < \varepsilon \ll 1 \tag{11}$$

and leave equations (8) unscaled, (this is suggested by the numerical results). We then look for a solution valid for small ε by expanding

$$\begin{aligned} f(\eta; \varepsilon) &= f_0(\eta) + \varepsilon^{1/2} f_1(\eta) + \varepsilon f_2(\eta) + \cdots\,, \\ h(\eta; \varepsilon) &= h_0(\eta) + \varepsilon^{1/2} h_1(\eta) + \varepsilon h_2(\eta) + \cdots\,. \end{aligned} \tag{12}$$

At leading order we obtain equations and boundary conditions which are essentially equations (8) the only differences being that n is replaced by n_0. At $O(\varepsilon^{1/2})$ we obtain the homogeneous problem

$$f_1''' + h_1 + f_0 f_1'' + f_0'' f_1 - 2\left(\frac{2n_0 + 3}{n_0 + 4}\right) f_0' f_1' = 0\,, \tag{13a}$$

$$\frac{1}{\sigma} h_1'' + f_0 h_1' + f_1 h_0' - \left(\frac{4n_0 + 1}{n_0 + 4}\right)(f_0' h_1 + f_1' h_0) = 0 \tag{13b}$$

subject to the boundary conditions

$$f_1(0)=f_1'(0)=h_1'(0)=0\,,\quad f_1'\to 0\,,\quad h_1\to 0\quad \text{as}\quad \eta\to\infty\,. \tag{13c}$$

Now equations (8) have a solution for all values of $n \geq n_0$ and for general values of n in this range the homogeneous system (13) has only a trivial solution. However, for just one specific value of n (i.e. at $n = n_0$) equations (13) do have a non-trivial solution and it is this requirement that determines the value of n_0. This has to be done numerically by taking $f_1''(0) = 1$ and then taking n_0 as one of the parameters to be adjusted in the boundary-value matching program. Reliable estimates for $\gamma = -0.5$ were supplied from the previous calculation and the solution could then be systematically advanced to different values of γ. The results are shown in Fig. 2. We can see from this figure that the value of n_0 depends on γ and that $n_0 \to -1$ as $\gamma \to 0^-$, while $n_0 \to -4$ as $\gamma \to -\infty$. Both of these limiting cases will be discussed below.

We now return to the problem of finding the solution for small ε noting that we would not expect the solution of equations (13) to have $f_1''(0) = 1$ and that the general solution $(\bar{f}_1, \bar{h}_1, \bar{\eta})$, with $\bar{f}_1''(0) = K$ (say), can be obtained from our particular solution (f_1, h_1, η) by putting

$$\bar{f}_1 = Kf_1\,,\quad \bar{h}_1 = Kh_1\,,\quad \bar{\eta} = \eta\,. \tag{14}$$

The equations at $O(\varepsilon)$ are, on using (14),

$$f_2''' + h_2 + f_0 f_2'' + f_2 f_0'' - 2\left(\frac{2n_0+3}{n_0+4}\right) f_0' f_2' = \frac{5}{(n_0+4)^2} f_0'^2 - K^2\left[\bar{f}_1 \bar{f}_1'' - \left(\frac{2n_0+3}{n_0+4}\right)\bar{f}_1'^2\right], \tag{15a}$$

$$\frac{1}{\sigma} h_2'' + f_0 h_2' + f_2 h_0' - \left(\frac{4n_0+1}{n_0+4}\right)(f_0' h_2 + f_2' h_0) = \frac{3}{(n_0+4)^2} f_0' h_0 - K^2\left[\bar{f}_1 \bar{h}_1' - \left(\frac{4n_0+1}{n_0+4}\right)\bar{f}_1' \bar{h}_1\right], \tag{15b}$$

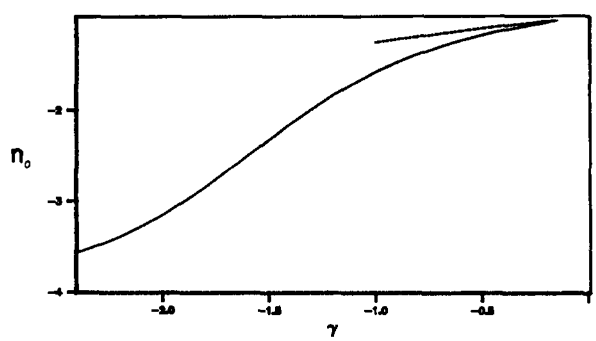

Fig. 2. A graph of n_0 plotted against γ (for $\sigma = 1.0$). The solution of equations (8) exists only for $n > n_0$. The expansion for small γ given by (26a) is shown by the broken line.

with boundary conditions

$$f_2(0) = f_2'(0) = h_2'(0) = 0\,, \quad f_2' \to 0\,, \quad h_2 \to 0 \quad \text{as} \quad \eta \to \infty\,. \tag{15c}$$

Now, to solve equations (15) numerically we have first to construct two particular integrals (f_a, h_a) and (f_b, h_b), which satisfy (15c) at $\eta = 0$ and have $f_i''(0) = 0$, $h_i(0) = 0$ $(i = a, b)$. For (f_a, h_a) we put $K = 0$ in equations (15a, b) and for (f_b, h_b) we put $K = 1$ and remove the first term on the right hand sides of these equations. We then construct two complementary functions (f_c, h_c) and (f_d, h_d), by taking $f_c''(0) = 1$, $h_c(0) = 0$ and $f_d''(0) = 0$, $h_d(0) = 1$. The full solution of equations (15) is then

$$f_2 = \lambda f_c + \mu f_d + f_a + K^2 f_b\,, \quad h_2 = \lambda h_c + \mu h_d + h_a + K^2 h_b\,, \tag{16}$$

for some constants λ and μ.

As $\eta \to \infty$, equations (15a, b) show that

$$h_i \to A_i\,, \quad f_i' \sim -\frac{A_i}{C_0}\eta + B_i \quad (i = 1, b, c, d)\,, \tag{17a}$$

where $C_0 = \lim_{\eta\to\infty} f_0(\eta)$, and to satisfy the boundary conditions as $\eta \to \infty$ we must have

$$\begin{aligned} &\lambda A_c + \mu A_d + A_a + K^2 A_b = 0\,, \\ &\lambda B_c + \mu B_d + B_a + K^2 B_b = 0\,. \end{aligned} \tag{17b}$$

However, the existence of a non-trivial solution to equations (13), which are the left hand sides of equations (15), guarantees the existence of a complementary function for these latter equations which satisfies all the required boundary conditions. Hence we must have

$$A_c B_d = A_d B_c\,, \tag{17c}$$

and it is then the requirement that equations (17b) are consistent that determines K, which, with (17c), gives

$$K^2 = \frac{B_a A_c - A_a B_c}{A_b B_c - B_b A_c}\,. \tag{17d}$$

We can now see why the solution cannot be continued past $n = n_0$. Consider a solution of equations (8) for a general value of $n > n_0$. A perturbation to this solution caused by making small changes in n will be a regular perturbation and the resulting equations, equivalent to the equations at $O(\varepsilon)$ in the above, will have a well-defined solution. This can be found from equations (17b) (with $K = 0$) which are now consistent since (17c) no longer holds. However at $n = n_0$ the equations at $O(\varepsilon)$ have a complementary function which satisfies all the required boundary conditions, in fact it was this requirement that determined n_0, and equations (17b), with $K = 0$, are now inconsistent. To remove this difficulty a singular perturbation, in powers of $\varepsilon^{1/2}$, is needed giving

$$\begin{aligned} &f''(0) = f_0''(0) + K(n - n_0)^{1/2} + \cdots\,, \\ &h(0) = h_0(0) + K h_1(0)(n - n_0)^{1/2} + \cdots\,, \end{aligned} \tag{18}$$

as $n \to n_0^+$ where K depends on γ and σ. (18) shows the singular nature of the solution at $n = n_0$.

We now consider the behaviour of n_0 as $\gamma \to 0^-$. A consideration of equations (8) suggests that we should put

$$n = -1 + n_1 |\gamma|^{5/4} + \cdots , \tag{19a}$$

with then

$$f = |\gamma|^{-1/4} F , \quad h = |\gamma|^{-1} H , \quad \zeta = |\gamma|^{-1/4} \eta , \tag{19b}$$

where n_1 is to be determined. We look for a solution by expanding

$$F(\zeta; \gamma) = F_0(\zeta) + |\gamma|^{5/4} F_1(\zeta) + \cdots ,$$

$$H(\zeta; \gamma) = H_0(\zeta) + |\gamma|^{5/4} H_1(\zeta) + \cdots . \tag{20}$$

At leading order we obtain the equations

$$F_0''' + H_0 + F_0 F_0'' - \frac{1}{3} F_0'^2 = 0 , \tag{21a}$$

$$\frac{1}{\sigma} H_0' + F_0 H_0 = 0 , \tag{21b}$$

where equation (21b) has been obtained by integrating once and applying the boundary conditions, which are

$$F_0(0) = 0 , \quad F_0'(0) = 0 , \quad H_0'(0) = 0 , \quad F_0' \to 0 , \quad H_0 \to 0 \quad \text{as} \quad \zeta \to \infty \tag{21c}$$

(primes now denote differentiation with respect to ζ). Equations (21) have arisen previously in [12] where it has been shown that they have a non-trivial solution $(\bar{F}_0, \bar{H}_0, \bar{\zeta})$, with $\bar{F}_0''(0) = 1$ and then, for $\sigma = 1$, $\bar{H}_0(0) = 0.87606$. The general solution, (F_0, H_0, ζ) with $F_o''(0) = L$ can be obtained from this particular solution by the transformation

$$F_0 = L^{1/3} \bar{F}_0 , \quad H_0 = L^{4/3} \bar{H}_0 , \quad \zeta = L^{-1/3} \bar{\zeta} . \tag{22}$$

At $O(|\gamma|^{5/4})$ we obtain the equations for $(\bar{F}_1, \bar{H}_1, \bar{\zeta})$, where $F_1 = \bar{F}_1$, $H_1 = L\bar{H}_1$ and $\bar{\zeta}$ is given by (22), namely

$$\bar{F}_1''' + \bar{H}_1 + \bar{F}_0 \bar{F}_0'' - \frac{2}{3} \bar{F}_0' \bar{F}_1' + \bar{F}_0'' \bar{F}_1 = \frac{5}{9} n_1 L^{1/3} \bar{F}_0'^2 , \tag{23a}$$

$$\frac{1}{\sigma} \bar{H}_1'' + \bar{F}_1 \bar{H}_0' + \bar{F}_0 \bar{H}_1' + \bar{F}_0' \bar{H}_1 + \bar{F}_1' \bar{H}_0 = \frac{5}{3} n_1 L^{1/3} \bar{F}_0' \bar{H}_0 , \tag{23b}$$

subject to the boundary conditions

$$\bar{F}_1(0) = 1 , \quad \bar{F}_1'(0) = 0 , \quad \bar{H}_1'(0) = -L^{-4/3} , \quad \bar{F}_1' \to 0 , \quad \bar{H}_1 \to 0 \quad \text{as} \quad \bar{\zeta} \to \infty . \tag{23c}$$

Equations (23) have a non-trivial complementary function, $\bar{F}_1 = \bar{\zeta} \bar{F}_0' + \bar{F}_0$, $\bar{H}_1 = \zeta \bar{H}_0' + 4\bar{H}_0$ which satisfies homogeneous boundary conditions at $\bar{\zeta} = 0$ and as $\bar{\zeta} \to \infty$ and hence to discuss the numerical solution of equations (23) we need only consider particular integrals. We construct three such particular integrals, namely (F_a, H_a) which has $F_a(0) = 0$, $H_a'(0) = 0$ and the term $n_1 L^{1/3}$ replaced by unity, (F_b, H_a) which has $F_b(0) = 1$, $H_b'(0) = 0$ and the right hand sides of equations (23) put to zero, and (F_c, H_c) which has $F_c(0) = 0$, $H_c'(0) = -1$ and again the right hand sides put to zero. Now, as $\bar{\zeta} \to \infty$

$$H_i \to C_i\,, \quad F_i' \sim -\frac{C_i\bar{\zeta}}{C_0} + D_1 \quad (i = a, b, c)\,, \tag{24a}$$

where $C_0 = \lim_{\bar{\zeta}\to\infty} \bar{F}_0$. Consequently to satisfy the boundary at $\bar{\zeta} \to \infty$ we must have

$$n_1 L^{1/3} C_a + C_b + L^{-4/3} C_c = 0\,, \tag{24b}$$

$$n_1 L^{1/3} D_a + D_b + L^{-4/3} D_c = 0\,. \tag{24c}$$

Equations (24b, c) now determine n_1 (and L), as

$$L^{-4/3} = \frac{C_b D_a - C_a D_b}{D_c C_a - C_c D_a}\,, \quad n_1 L^{1/3} = \frac{D_b C_c - C_b D_c}{C_a D_c - D_a C_c}\,. \tag{25}$$

Numerical integrations of equations (23) give, for $\sigma = 1$, $C_a = 0.841484$, $D_a = 0.878057$; $C_b = 0.605311$, $D_b = -1.576148$; $C_c = -0.690947$, $D_c = 3.811802$, leading to $L = 1.71518$ and $n_1 = -0.26683$, from which it follows that, for $\sigma = 1$,

$$n_0 \sim -1 - 0.26683|\gamma|^{5/4} + \cdots \quad \text{as} \quad \gamma \to 0^-\,. \tag{26a}$$

Furthermore, from (19b) and (22) we have that, for $\sigma = 1$,

$$h(0) \sim 1.79864|\gamma|^{-1} + \cdots\,, \quad f''(0) = 1.71518|\gamma|^{-3/4} + \cdots \tag{26b}$$

as $\gamma \to 0^-$. A graph of n_0 obtained from (26a) is also shown in Fig. 2 (by the broken line) where we can see that it is in good agreement with the computed values for γ small.

Finally, it remains, in the present context, to consider the nature of the singularity in the solution as $n \to -1$ for blowing (i.e. $\gamma > 0$). This is what we treat in the next section.

(b) Solution as $n \to -1$ for $\gamma > 0$

To discuss this case we put

$$n = -1 + \delta\,, \quad 0 < \delta \ll 1 \tag{27}$$

and take γ to be of $0(1)$. A consideration of equations (8) then suggests that we put

$$f = \delta^{-1} F\,, \quad h = \delta^{-4} H\,, \quad \tau = \delta^{-1}\eta \tag{28a}$$

and look for a solution by expanding

$$F(\tau; \delta) = F_0(\tau) + \delta F_1(\tau) + \cdots\,, \tag{28b}$$

$$H(\tau; \delta) = H_0(\tau) + \delta H_1(\tau) + \cdots\,. \tag{28c}$$

At leading order we obtain the system given by equations (21) and boundary conditions (22), now taking the general solution to have $F_0''(0) = M$ (say). At $O(\delta)$ we obtain equations (23a, b) but now with $n_1 = 1$ (and L replaced by M) and subject to the boundary conditions

$$\bar{F}_1(0) = -\gamma\,, \quad \bar{F}_1'(0) = 0\,, \quad \bar{H}_1'(0) = 0\,, \quad \bar{F}_1' \to 0\,, \quad \bar{H}_1 \to 0 \quad \text{as} \quad \tau \to \infty\,. \tag{29}$$

The numerical solution of this problem then requires only the two particular integrals (F_a, H_a) and (F_b, H_b) with the condition as $\tau \to \infty$ giving

$$M^{1/3}C_a - \gamma C_b = 0\,, \quad M = \left(\frac{\gamma C_b}{C_a}\right)^3\,. \tag{30}$$

From the values given previously we find that $M = 0.37222\gamma^3$ for $\sigma = 1$. It then follows that

$$h(0) \sim 0.23457\gamma^4(n+1)^{-4} + \cdots\,, \tag{31a}$$

$$f''(0) \sim 0.37222\gamma^3(n+1)^{-3} + \cdots\,, \tag{31b}$$

as $n \to -1$ from above.

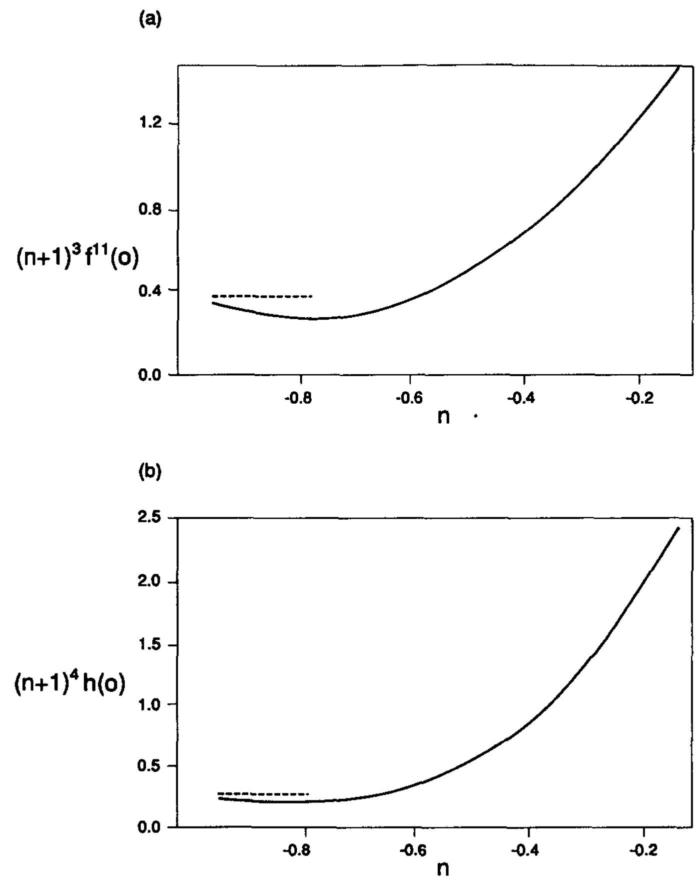

Fig. 3. Graphs of (a) $(n+1)^3 f''(0)$, (*b*) $(n+1)^4 h(0)$ plotted against n for blowing $\gamma = 1.0$ (shown by the full lines). The broken lines are the asymptotic values given by equation (31).

As a check on the above analysis we solved equations (8) numerically for $\gamma = 1.0$ (and $\sigma = 1$) for decreasing values of n. The results are shown in Fig. 3 where we plot $(n+1)^3 f''(0)$ and $(n+1)^4 h(0)$ against n (shown by the full line). The broken lines show the appropriate asymptotic values given by (31). These graphs show that these asymptotic values are approached as $n \to -1$ and acts as a confirmation of our theory.

The asymptotic forms (31) will not hold as $\gamma \to 0$ where the behaviour for the case $\gamma = 0$, as given by Merkin [11], will be recovered. In [11] it was shown that $f''(0)$ is of $O((n+1)^{-3/5})$ and $h(0)$ is of $O((n+1)^{-4/5})$ as $n \to -1$. From (31) it then follows that these forms will hold when γ is of $O((n+1)^{4/5})$ as $n \to -1$, in line with the results given previously in (26). A solution can then be developed for both $(n+1)$ and γ small using these scalings. This follows very closely those already described and is not pursued further here.

We can get further insight into the nature of the solutions of equations (8) by considering their behaviour as $|\gamma| \to \infty$, and this is what we now do.

(c) Solution for strong suction

Here we obtain a solution of equations (8) for $\gamma < 0$ valid as $|\gamma| \to \infty$. The solutions for strong suction for the prescribed wall temperature case [3, 5] suggest that we put

$$f = \gamma + |\gamma|^{-4} S, \quad h = |\gamma|^{-1} T, \quad \xi = |\gamma| \eta. \tag{32}$$

On substituting (32) into equations (8), we find that, at leading order

$$T = \frac{1}{\sigma} e^{-\sigma\xi}, \tag{33a}$$

$$S = \frac{1}{\sigma^3(\sigma - 1)} (e^{-\sigma\xi} - \sigma e^{-\xi} + \sigma - 1), \quad (\sigma \neq 1), \tag{33b}$$

$$= 1 - (\xi + 1) e^{-\xi}, \quad (\sigma = 1). \tag{33c}$$

(33c) can be obtained directly from the equations or by letting $\sigma \to 1$ in (33b).

From (32) and (33) it then follows that the boundary layer becomes thin, with a thickness of $O(|\gamma|^{-1})$ and that

$$\begin{aligned} h(0) &\sim \sigma^{-1}|\gamma|^{-1}(1 + O(|\gamma|^{-5})), \\ f''(0) &\sim \sigma^{-2}|\gamma|^{-2}(1 + O(|\gamma|^{-5})), \\ f(\infty) &\sim -\gamma + \sigma^{-3}|\gamma|^{-4}(1 + O|\gamma|^{-5}), \end{aligned} \tag{34}$$

as $|\gamma| \to \infty$. This asymptotic behaviour can be seen in Fig. 4, where we plot values of $f''(0)$, $f(\infty)$ and $h(0)$ against γ (for $n = 1$ and $\sigma = 1$) obtained from a numerical solution of equations (8) (shown by the full lines). The asymptotic solution for $|\gamma|$ large as given by (34) is also shown (by the broken lines). These figures show that these asymptotic forms are approached rapidly as $|\gamma|$ is increased, and are in close agreement with the numerically determined values from $\gamma \approx -1.6$. This is, perhaps to be expected as the error in the leading order terms is of $O(|\gamma|^{-5})$.

It is worth noting that expressions (33) are independent of the value of the exponent n. This enters the terms of order $O(|\gamma|^{-5})$ in expansion (34). Consequently, equations (8) will

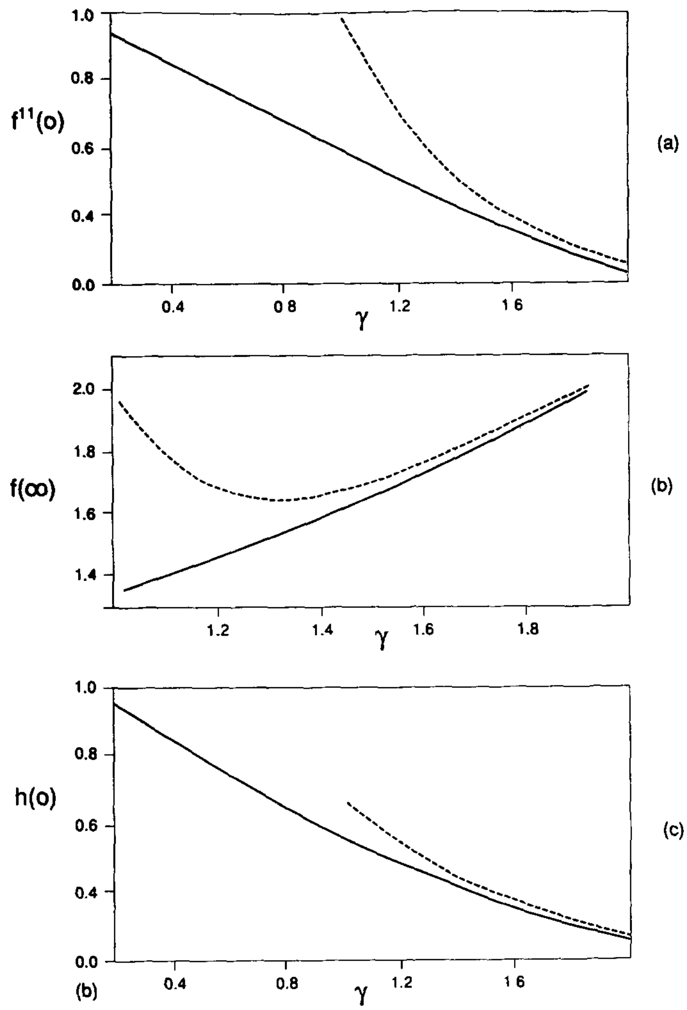

Fig. 4. Graphs of (a) $f''(0)$, (b) $f(\infty)$, (c) $h(0)$ plotted against $|\gamma|$ for suction, with $n = 1$, $\sigma = 1$. The asymptotic forms (34) are shown by the broken lines.

have a solution for all n (provided only that $n > -4$, the condition required for forward boundary layers) in the limit as $|\gamma| \to \infty$. This is in line with the results shown in Fig. 2.

(d) Solution for strong blowing

Here we obtain a solution of equations (8) for $\gamma > 0$ valid as $\gamma \to \infty$ and where we assume, from the discussion in the previous section, that $n > -1$. The numerical solutions of these equations show that both the skin friction $f''(0)$ and wall temperature $h(0)$ increase and that the boundary layer thickens as γ is increased. To get the initial scalings for this limit we start by noting that f is of $O(\gamma)$ and that we expect the leading order terms to be given by the inviscid versions of equations (8). Now, suppose that h is of $O(\gamma^r)$ and the boundary layer has thickness of $O(\gamma^s)$ as $\gamma \to \infty$, for some positive exponents r and s to be determined. Then a balancing of the inviscid (convective) terms, of $O(\gamma^{2-2s})$, and buoyancy force term, $O(\gamma^r)$, gives $r = 2 - 2s$. Also, the viscous terms are of $O(\gamma^{-(1+s)})$ relative to the convective terms, and $h'(0)$ is of $O(\gamma^{s-r})$. Now, when we come to solve the transformed equations we find that we must have $h'(0)=0$ at leading order (so $r>s$) and that the perturbations to the leading order solution from $h'(0)$ and from the viscous terms should be of the same order. This gives a further relation between r and s, namely $r=2s+1$, then giving $r=3/2$, $s=\frac{1}{4}$.

The above discussion suggests that we start by putting

$$f=-\gamma\phi\,,\quad h=\gamma^{3/2}g\,,\quad y=\gamma^{-1/4}Y\,. \tag{35}$$

Equations (8) become

$$\phi\phi''-\alpha\phi'^2+g-\gamma^{-5/4}\phi'''=0\,, \tag{36a}$$

$$\phi g'-\beta\phi' g-\frac{1}{\sigma}\gamma^{-5/4}g''=0\,, \tag{36b}$$

where primes denote differentiation with respect to y and where, for ease in the discussion to follow, we have put $\alpha=(2n+3)/(n+4)$ and $\beta=(4n+1)/(n+4)$. The boundary conditions to be satisfied are

$$\phi(0)=1\,,\quad \phi'(0)=0\,,\quad g'(0)=-\gamma^{-5/4}\,. \tag{36c}$$

The outer boundary conditions are relaxed at this stage.

Equations (36) suggest that we look for a solution for γ large by expanding

$$\phi(y;\gamma)=\phi_0(y)+\gamma^{-5/4}\phi_1(y)+\cdots\,,$$

$$g(y;\gamma)=g_0(y)+\gamma^{-5/4}g_1(y)+\cdots\,. \tag{37}$$

At leading order we obtain from equation (36b)

$$\phi_0 g_0'-\beta\phi_0' g_0=0\,. \tag{38a}$$

The solution of equation (38a) subject to the condition, from (36c), that $g_0'(0)=0$ is

$$g_0=k\phi_0^{\beta} \tag{38b}$$

for some positive constant k to be determined. We note that (38b) satisfies $g_0'(0)=0$, and that the condition $h'(0)=-1$ cannot be satisfied at this order. Equation (36a) then gives

$$\phi_0\phi_0''-\alpha\phi_0'^2+k\phi_0^{\beta}=0\,, \tag{39a}$$

subject to

$$\phi_0(0)=1\,,\quad \phi_0'(0)=0\,. \tag{39b}$$

Equation (39) can be solved implicitly to get

$$\phi_0'=-\left(\frac{2k}{2\alpha-\beta}\right)^{1/2}(\phi_0^{\beta}-\phi_0^{2\alpha})^{1/2}\,, \tag{40a}$$

and then

$$\left(\frac{2k}{2\alpha-\beta}\right)^{1/2} y = \int_{\phi_0}^{1} \frac{dt}{(t^{\beta}-t^{2\alpha})^{1/2}} . \tag{40b}$$

Note that $2\alpha - \beta = 5/(n+4) > 0$.

Since $2\alpha > \beta$, (40a) shows that $\phi_0' \leq 0$ on $0 \leq \phi_0 \leq 1$ and hence ϕ_0 is a monotone decreasing function of y in this interval, and that $\phi_0' = 0$ at $\phi_0 = 0$ (as well as at $\phi_0 = 1$). Furthermore, the integrand in (40b) is of $O(t^{-\beta/2})$ as $t \to 0$ and the integral will exist (as an improper integral) as $\phi_0 \to 0$ provided $1 - \beta/2 > 0$, i.e. $n < 7/2$. Alternatively, this result shows that ϕ_0 will be zero at a finite value of y (y_0, say) for $n < 7/2$, whereas for $n \geq 7/2$, $\phi_0 \to 0$ only as $y \to \infty$. For $n = 7/2$ ($\alpha = 4/3$, $\beta = 2$) equation (40b) yields the specific solution

$$\phi_0 = \operatorname{sech}^3\left(\sqrt{\frac{k}{3}}\, y\right), \quad g_0 = k \operatorname{sech}^6\left(\sqrt{\frac{k}{3}}\, y\right), \tag{41a}$$

and then

$$\phi_0 \sim 8 \exp(-\sqrt{3k}\, y) \quad \text{as} \quad y \to \infty . \tag{41b}$$

For $n > 7/2$ the decay as $y \to \infty$ is only algebraic, with

$$\phi_0 \sim \left(\frac{k(\beta-2)^2}{2(2\alpha-\beta)}\right)^{1/(2-\beta)} y^{2/(2-\beta)} \quad \text{as} \quad y \to \infty . \tag{42}$$

For $n < 7/2$, equation (40b) shows that $\phi_0 = 0$ at $y = y_0$ where

$$y_0 = \left(\frac{2\alpha-\beta}{k}\right)^{1/2} \int_0^1 \frac{dt}{(t^{\beta}-t^{2\alpha})^{1/2}} . \tag{43a}$$

This integral can be evaluated in terms of Beta functions [13], as

$$y_0 = \left(\frac{(n+4)\pi}{10k}\right)^{1/2} \frac{\left(-\frac{2n+3}{10}\right)!}{\left(\frac{1-n}{5}\right)!}, \quad n < \frac{7}{2} . \tag{43b}$$

This completes the discussion of the leading order solution and we can now turn to the terms of $O(\gamma^{-5/4})$ in expansion (37). We find that we need only consider the equation for g_1 at this stage, which is

$$\phi_0 g_1' - \beta\phi_0' g_1 + \phi_1 g_0' - \beta\phi_1' g_0 = \frac{1}{\sigma} g_0'' , \tag{44a}$$

subject to

$$g_1'(0) = -1 . \tag{44b}$$

A solution to equation (44a) can be expressed as

$$g_1 = k\beta\phi_0^{\beta-1}\phi_1 + A_1\phi_0^\beta + \frac{k^2\beta\phi_0^\beta}{(2\alpha-\beta)\sigma}\int_0^y \frac{(3\beta-2)\phi_0^\beta - 2(\alpha+\beta-1)\phi_0^{2\alpha}}{\phi_0^3}\, d\bar{s} \tag{45}$$

for some constant A_1.

It is the application of boundary condition (44b) that then determines k as

$$k = \left(\frac{\sigma}{\beta}\right)^{1/2}, \quad \beta > 0 . \tag{46}$$

We can now see why the perturbations to the leading order solution from the viscous terms and from the heat flux boundary condition have to be of the same order. The final term in expression (45) comes from the viscous term in equation (44a) and without this term the boundary condition (44b) could not be satisfied. Conversely, the inclusion of this term, but now trying to apply the boundary condition $g_1'(0) = 0$ also leads to an inconsistency. Finally, we note that (46) holds only for $\beta > 0$, i.e. $n > -\frac{1}{4}$, and the case $-1 < n < -\frac{1}{4}$ will require further discussion. We can also see from (38b) that a problem arises for n in this range, since we must have $\beta > 0$ to ensure that g_0 remains bounded as $\phi_0 \to 0$.

The above discussion shows that we have three separate cases to consider, namely n in the ranges $-1 < n < -\frac{1}{4}$, $-\frac{1}{4} < n < \frac{7}{2}$, $n > \frac{7}{2}$. We start by considering n in the range $-\frac{1}{4} < n < \frac{7}{2}$, i.e. $0 < \beta < 2$. Here, from (35), (38b) (40a) and (46), we have that

$$f''(0) \sim \left(\frac{\sigma}{\beta}\right)^{1/2} \gamma^{1/2} + \cdots, \quad h(0) \sim \left(\frac{\sigma}{\beta}\right)^{1/2} \gamma^{3/2} + \cdots, \tag{47}$$

as $\gamma \to \infty$. Graphs of $f''(0)$ and $h(0)$ calculated from a numerical solution of equations (8) for $n = 1$ and $\sigma = 1$ are shown in Fig. 5 (by the full lines). Also shown in this figure (by the broken lines) are the asymptotic forms as given by (47). We can see that the two sets of curves are approaching each other as γ is increased, though the convergence is slow (the error is from (37), of $O(\gamma^{-5/4})$). The agreement was seen to improve at higher values of γ (not shown on these graphs). We note that, for $n = 1$ ($\alpha = \beta = 1$) equations (38b) and (40) give the simple solution.

$$\phi_0 = \frac{1}{2}(1 + \cos(\sqrt{2k}\, y)) , \quad g_0 = \frac{k}{2}(1 + \cos(\sqrt{2k}\, y)) , \tag{48a}$$

with (46) giving $k = \sigma^{1/2}$, and (48a) giving

$$y_0 = (2\sqrt{\sigma})^{-1/2}\pi , \tag{48b}$$

in agreement with (43b).

In this case the outer boundary conditions are satisfied at a finite value of y, the point y_0 given by (43b), and not smoothly as $y \to \infty$. Thus an outer region is required, which will be thin relative to the inner region (given by (35)) and in which the viscous terms will be important at leading order. To get the scalings for this outer region we first have to determine the forms of ϕ_0 and g_0 as $y \to y_0$. A little calculation reveals that

$$\phi_0 \sim \mu(y_0 - y)^{2/(2-\beta)} , \quad g_0 \sim k\mu^\beta (y_0 - y)^{2\beta/(2-\beta)} , \tag{49}$$

as $y \to y_0$, where the constant $\mu = [(2k/2\alpha - \beta)^{1/2}(2 - \beta/2)]^{2/(2-\beta)}$ (with k given by (46)). For $n = 1$ (49) reduces to $\phi_0 \sim k/2\,(y_0 - y)^2$ as can be calculated directly from (48).

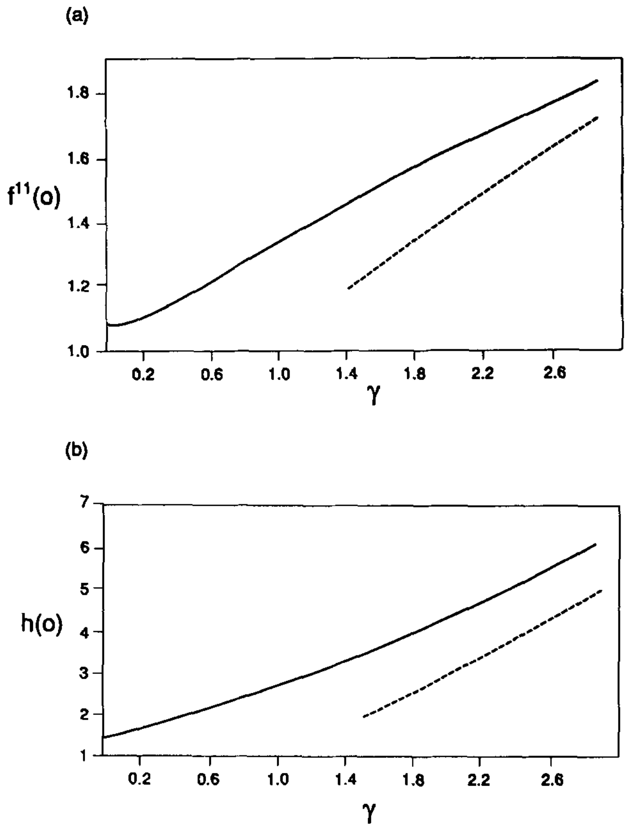

Fig. 5. Graphs of (a) $f''(0)$, (b) $h(0)$ plotted against γ for blowing with $n = 1$, $\sigma = 1$. The asymptotic forms given by (47) are shown by the broken lines.

To get the scalings for this outer region we note that the solution must match with (49) at its inner edge, and that all the terms in equations (8) must balance. This leads us to write

$$f = \gamma^{(2-n)/6}\Phi\,, \quad h = \gamma^{2(2-n)/3}G\,, \quad \eta = \gamma^{1/4}y_0 + \gamma^{(n-2)/6}Y\,. \tag{50}$$

Substituting (50) into equations (8) gives

$$\Phi''' + G + \Phi\Phi'' - \alpha\Phi'^2 = 0\,, \tag{51a}$$

$$\frac{1}{\sigma}G'' + \Phi G' - \beta\Phi' G = 0\,, \tag{51b}$$

where primes now denote differentiation with respect to Y. The boundary conditions are that

$$\Phi' \to 0\,, \quad G \to 0 \quad \text{as} \quad Y \to \infty\,, \tag{51c}$$

and, from (49) on matching with the inner region, that

$$\Phi \sim -\mu(-Y)^{2/(2-\beta)}\,, \quad G \sim k\mu^{\beta}(-Y)^{2\beta/(2-\beta)} \quad \text{as} \quad Y \to -\infty\,. \tag{51d}$$

The solution of the shear layer equations (51a, b) subject to boundary conditions (51c) and matching conditions (51d) then completes the solution (at least to leading order).

It is worth noting that the scalings for this outer shear layer, given by (50), depend on the exponent n. This is somewhat unusual and did not occur in the prescribed wall temperature case [5]. Also, this shear layer has a thickness of $O(\gamma^{(n-2)/6})$ which is thin relative to the inner inviscid region, of thickness $O(\gamma^{1/4})$, only for $n<\frac{7}{2}$ which is the case we are considering. For $n=\frac{7}{2}$ (i.e. $\beta=2$) the solution is given by (41) with k given by (46), and has exponential decay as $y\to\infty$. A consideration of the higher order terms in the expansion suggests that these will also all have exponential decay at infinity and hence the complete solution is given by the solution in the inner region, with no further outer region being required.

Now consider the case $n>\frac{7}{2}$ ($\beta>2$). Here the solution in the inner region is still given by (40) but now $y_0\to\infty$ and the behaviour of ϕ_0 (and hence g_0) for large y is given by (42), with k given by (46). The decay is now only algebraic and we expect [14] that the higher order terms in the expansion will lead to algebraic growth at large y. We find this to be the case and a further outer region is required to achieve the outer boundary conditions (through exponential decay). The thickness of this outer region will be large in relation to the inner region and the scalings for it are essentially the same as (50), though the independent variable Y is not now centred on y_0. Thus, in this case, we write

$$f=\gamma^{(2-n)/6}\Phi\,,\quad h=\gamma^{2(2-n)/3}G\,,\quad \eta=\gamma^{(n-2)/6}Y\,. \tag{52a}$$

Substituting (52) into equations (8) leads again to equations (51a, b) to be solved subject to boundary conditions (51c). However, the matching with the inner region, now leads to the inner boundary conditions that

$$\Phi=-\mu Y^{-2/(\beta-2)}+\cdots\,,\quad G=-k\mu^{\beta}Y^{-2\beta/(\beta-2)}+\cdots\quad \text{as}\quad Y\to 0\,, \tag{52b}$$

with μ as defined above. The solution of this problem given by equations (51a, b, c) and (52b) then completes the solution to leading order. Before leaving this case we note that the thickness of the outer region is still of $O(\gamma^{(n-2)/6})$ but this is now large relative to the thickness of the inner region.

Finally we have to consider the case $-1<n<-\frac{1}{4}$. Here $\beta<0$ and the solutions given previously for $n>-\frac{1}{4}$ cannot apply and a new structure is required. Numerical solutions of equations (8) for $n=-\frac{1}{2}$ show that the boundary thickness decreases as γ is increased in contrast to the previous cases where it increased with increasing γ. This suggests that the viscous terms will be important to leading order. To obtain the scalings for this case, we will still have f of $O(\gamma)$. If we again assume that h is of $O(\gamma^r)$ and that the boundary layer has a thickness of $O(\gamma^s)$ then a balancing of all the terms in equation (8a) gives $1-3s=r=2-2s$, i.e. $s=-1$, $r=4$, so that the boundary layer has a thickness of $O(\gamma^{-1})$ and the wall temperature is large, of $O(\gamma^4)$.

The above suggests that we put

$$f=\gamma\phi\,,\quad h=\gamma^4 g\,,\quad y=\gamma\eta\,, \tag{53}$$

with equations (8) becoming

$$\phi'''+g+\phi\phi''-\alpha\phi'^2=0\,, \tag{54a}$$

$$\frac{1}{\sigma} g'' + \phi g' - \beta \phi' g = 0 \,, \tag{54b}$$

subject to the boundary conditions

$$\phi(0) = -1 \,, \quad \phi'(0) = 0 \,, \quad g'(0) = -\gamma^{-5} \,, \quad \phi' \to 0 \,, \quad g \to 0 \quad \text{as} \quad y \to \infty \,. \tag{54c}$$

Conditions (54c) suggest that we look for a solution by expanding in powers of γ^{-5}. The leading order terms satisfy equations (54a, b) and conditions (54c) with $g'(0) = 0$. A numerical solution gives, for $\sigma = 1$, $g(0) = 1.10544$ and $\phi''(0) = 0.83005$. Hence,

$$f''(0) \sim 0.83005\gamma^3 + \cdots \,, \quad h(0) \sim 1.10544\gamma^4 + \cdots \quad \text{as} \quad \gamma \to \infty \,. \tag{55}$$

We solved equations (8) numerically for $n = -\frac{1}{2}$ (and $\sigma = 1$) for increasing γ. The results are shown in Fig. 6 (by the full lines) where we plot $\gamma^{-3} f''(0)$ and $\gamma^{-4} h(0)$ against γ. These curves can be seen to approach the constant values (shown by the broken lines) as suggested by (55) for increasing γ and confirms the above theory. Note that the approach to this asymptotic solution is much more rapid than was previously seen for $n = 1$. This is to be expected as here the error is much smaller, $O(\gamma^{-5})$.

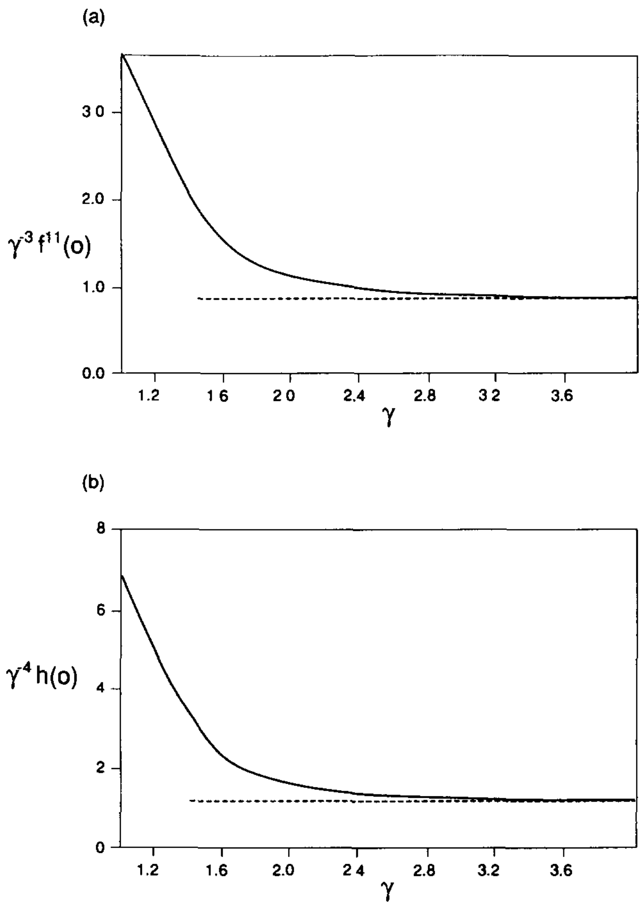

Fig. 6. Graphs of (a) $\gamma^{-3} f''(0)$, (b) $\gamma^{-4} h(0)$ plotted against γ for blowing with $n = -0.5$ and $\sigma = 1$. The asymptotic forms given by (55) are shown by the broken lines.

We have now completed our discussion of the similarity equations (8) and we consider next the case of uniform wall heat flux with a uniform transpiration velocity.

4. Uniform wall conditions

Here we obtain a solution of equations (4) subject to the boundary conditions (5b) and that

$$v = \pm 1 , \quad u = 0 , \quad \frac{\partial \theta}{\partial y} = -1 \quad \text{on} \quad y = 0 \tag{56}$$

(throughout this section the upper sign will be taken for blowing and the lower sign for suction). This problem does not have a similarity solution and to obtain a solution valid for all $x \geqslant 0$ the governing equations have to be solved numerically. The first step in obtaining such a numerical solution is to make a transformation of equations (4) appropriate to the solution for x small. To do this we put

$$\psi = \mp x + x^{4/5} f(x, \eta) , \quad \theta = x^{1/5} h(x, \eta) , \quad \eta = y / x^{1/5} . \tag{57}$$

Equations (4) become

$$\frac{\partial^3 f}{\partial \eta^3} + h + \left(\frac{4}{5} f \mp x^{1/5}\right) \frac{\partial^2 f}{\partial \eta^2} - \frac{3}{5}\left(\frac{\partial f}{\partial \eta}\right)^2 = x\left(\frac{\partial f}{\partial \eta} \frac{\partial^2 f}{\partial \eta \, \partial x} - \frac{\partial f}{\partial x} \frac{\partial^2 f}{\partial \eta^2}\right) , \tag{58a}$$

$$\frac{1}{\sigma} \frac{\partial^2 h}{\partial \eta^2} + \left(\frac{4}{5} f \mp x^{1/5}\right) \frac{\partial h}{\partial \eta} - \frac{1}{5} h \frac{\partial f}{\partial \eta} = x\left(\frac{\partial f}{\partial \eta} \frac{\partial h}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial h}{\partial \eta}\right) , \tag{58b}$$

subject to the boundary conditions

$$f = 0 , \quad \frac{\partial f}{\partial \eta} = 0 , \quad \frac{\partial h}{\partial \eta} = -1 \quad \text{on} \quad \eta = 0 , \quad \frac{\partial f}{\partial \eta} \to 0 , \quad h \to 0 \quad \text{as} \quad \eta \to \infty . \tag{58c}$$

Equations (58) show that the behaviour for small x is given by the solution for uniform wall heat flux without transpiration, given previously by Sparrow and Gregg [15]. Transpiration effects give an $O(x^{1/5})$ correction to this and lead to an expansion in powers of $x^{1/5}$. This is straightforward to obtain and is not pursued further, except to note that to develop the numerical solution from $x = 0$ we must use a streamwise variable $\xi = x^{1/5}$, with equations (58) being transformed accordingly, to accommodate the fact that derivatives with respect to x become infinite as $x \to 0$.

The numerical method we used is essentially the same method that we have used successfully on other similar boundary-layer problems and is described in a little detail in [16]. We started the solution from $x = 0$ using equations (58) (with x replaced by ξ) and integrated forwards until $x = 1$ ($\xi = 1$), where we reverted to solving the original equations (4). The velocity and temperature profiles calculated at $x = 1$ from equations (58) were used as starting profiles for the solution of equations (4) for $x > 1$ and thus a smooth transition from one solution regime to the other was achieved. We decided to adopt this strategy rather than use a continuous transformation of variables, as suggested by Hunt and Wilks [17], in order that the behaviour for large x might be more clearly seen, and so as not to impose any particular form of behaviour on the solution for large x.

(a) Suction

Consider first the case of suction. The results obtained from the numerical integration of equations (4) and (58) are shown in Fig. 7 where we give plots of the skin friction parameters $\tau_\omega = (\partial u/\partial y)_{y=0}$ and the wall temperature $\theta_\omega = \theta(x, 0)$ against x (for $\sigma = 1$). We can see from this figure that $\theta_\omega \to 1$ very rapidly, and that $\tau_\omega \to 1$, slightly more slowly, as x is increased. Also, the numerical results showed that the solution became uniform in y for large x, with the boundary layer having a constant thickness and the velocity and temperature profiles being dependent only on y.

To examine this behaviour in more detail we put

$$\psi(x, y) = x + \Psi_0(y) + \phi(x, y)\,, \quad \theta = \theta_0(y) + h(x, y)\,, \tag{59}$$

where $\phi(x, y)$ and $h(x, y)$ are small for x large. Equations (4) give

$$\Psi_0''' + \theta_0 + \Psi_0'' = 0\,, \tag{60a}$$

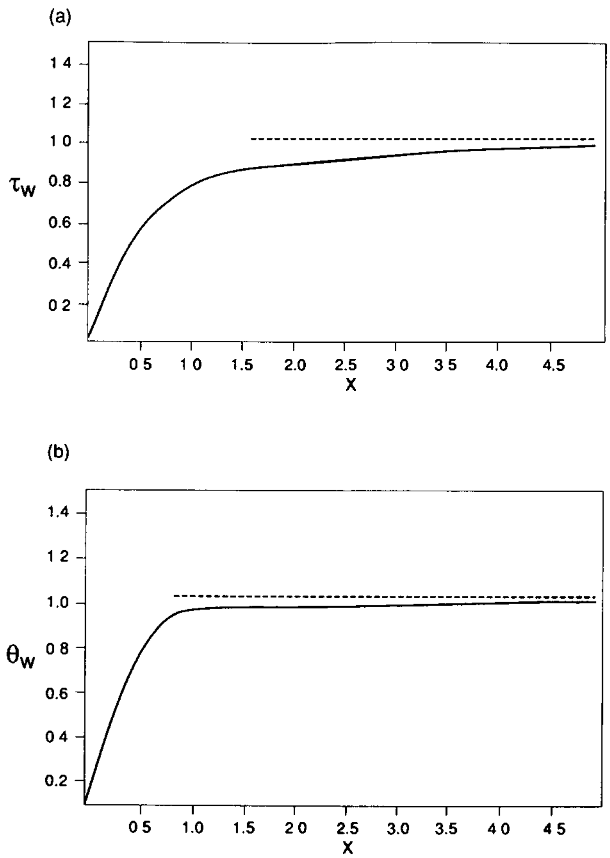

Fig. 7. Graphs of (a) skin friction τ_ω, (b) wall temperature θ_ω, for uniform suction and uniform wall heat flux, with $\sigma = 1$, obtained from a numerical solution of equations (4) and (58). The asymptotic solution is shown by the broken line.

$$\frac{1}{\sigma}\theta_0'' + \theta_0 = 0 , \tag{60b}$$

which have solution, satisfying

$$\Psi_0(0) = 0 , \quad \Psi_0'(0) = 0 , \quad \theta_0'(0) = -1 , \quad \Psi_0' \to 0 , \quad \theta_0 \to 0 \quad \text{as} \quad y \to \infty , \tag{60c}$$

as

$$\theta_0 = \sigma^{-1} \mathrm{e}^{-\sigma y} , \tag{61a}$$

$$\Psi_0 = \frac{1}{\sigma^3(\sigma - 1)} [\mathrm{e}^{-\sigma y} - \sigma \, \mathrm{e}^{-\sigma y} + \sigma - 1] \quad (\sigma \neq 1) , \tag{61b}$$

$$= 1 - \mathrm{e}^{-y} - y \, \mathrm{e}^{-y} \quad (\sigma = 1) . \tag{61c}$$

From (61) we see that

$$\theta_\omega \to \frac{1}{\sigma} , \quad \tau_\omega \to \frac{1}{\sigma^2} \quad \text{as} \quad x \to \infty , \tag{62}$$

and these are the results shown in Fig. 7.

Next we consider the equations for $\phi(x, y)$ and $h(x, y)$. Since these are small for x large products of terms involving ϕ and h can be neglected to leading order. This leads to linear equations and we look for a solution of these by putting

$$\phi(x, y) = \mathrm{e}^{-\lambda x}\Phi(y) , \quad h(x, y) = \mathrm{e}^{-\lambda x}H(y) . \tag{63}$$

The eigenvalue λ is then found from the system

$$\Phi''' + H + \Phi'' + \lambda(\Psi_0'\Phi' - \Psi_0''\Phi) = 0 , \tag{64a}$$

$$\frac{1}{\sigma}H'' + H' + \lambda(\Psi_0'H - \theta_0'\Phi) - 0 , \tag{64b}$$

with boundary conditions

$$\Phi(0) = 0 , \quad \Phi'(0) = 0 , \quad H'(0) = 0 , \quad \Phi' \to 0 , \quad H \to 0 \quad \text{as} \quad y \to \infty . \tag{64c}$$

The form of solution (63) could also involve a power of x, see, for example, Stewartson [18], though the dominant behaviour will be given by the exponential.

It is readily seen that the eigenfunctions ϕ and h cannot be an inverse power of x multiplied by the corresponding functions of y. For if this were the case then the equations for Φ and H would be (64a, b) with the terms involving λ put to zero, and these cannot have a non-trivial solution satisfying boundary conditions (64c).

The problem then is to find the largest value of λ which gives a non-trivial solution to equations (64). This had to be done numerically, forcing a non-trivial solution by taking $\Phi''(0) = 1$. The resulting eigenvalues λ were seen to depend on the Prandtl number σ and are shown in Fig. 8. For $\sigma = 1$, we found $\lambda = 0.40048$, and as σ is increased the largest eigenvalue also increases rapidly, while for $\sigma < 1$, it decreases rapidly. This shows that the

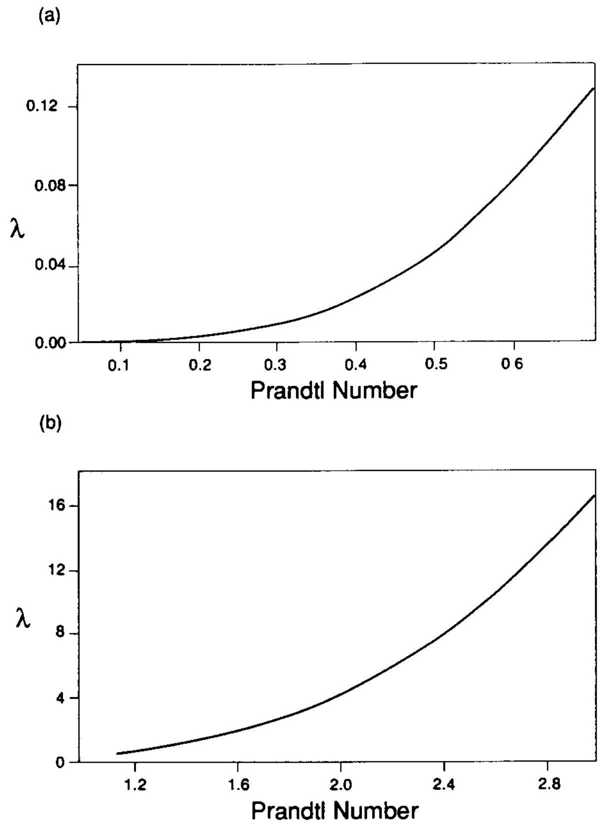

Fig. 8. The largest eigenvalues λ calculated from equations (64) plotted against σ, (a) $\sigma \leq 1$, (b) $\sigma \geq 1$.

rate of approach to the asymptotic profiles (61), which is of $O(e^{-\lambda(\sigma)x})$ for x large, will increase rapidly with increasing Prandtl number.

(b) Blowing

The results obtained from solving equations (4) and (58) for the case of blowing through the wall are shown in Figs. 9 and 10. In Fig. 9 we give graphs of τ_ω and θ_ω for small x. The singular behaviour as $x \to 0$ in τ_ω, of $O(x^{2/5})$, and θ_ω, of $O(x^{1/5})$, given by (57), can clearly be seen in these graphs. In Fig. 10 we give plots of τ_ω and θ_ω for larger values of x; note that in these figures they are plotted against $\log x$. Figure 10 shows that both the skin friction and the wall temperature increase with x. The numerical results also show that the boundary-layer thickness increases with x and that the velocity and temperature profiles develop a two-layer structure with there being a thick inner region next to the wall made up chiefly of fluid blown through the wall and a much thinner outer shear layer at the outer edge of which the ambient conditions are attained.

To complete the discussion we derive a solution of equations (4) valid for large x. We start by obtaining the scalings for the inner region, in which we assume that the temperature is of $O(x^r)$ and the boundary layer has a thickness of $O(x^s)$, for some positive exponents r and s to be found. Now, in this inner region which will be basically inviscid, the stream function ψ is of $O(x)$, with a balancing of convective and buoyancy force terms then giving $r = 1 - 2s$.

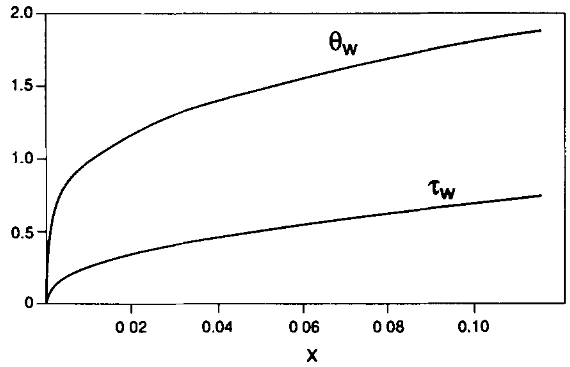

Fig. 9. Graphs of skin friction τ_ω and wall temperature θ_ω for small x obtained from the numerical solution of equations (58) with $\sigma = 1$.

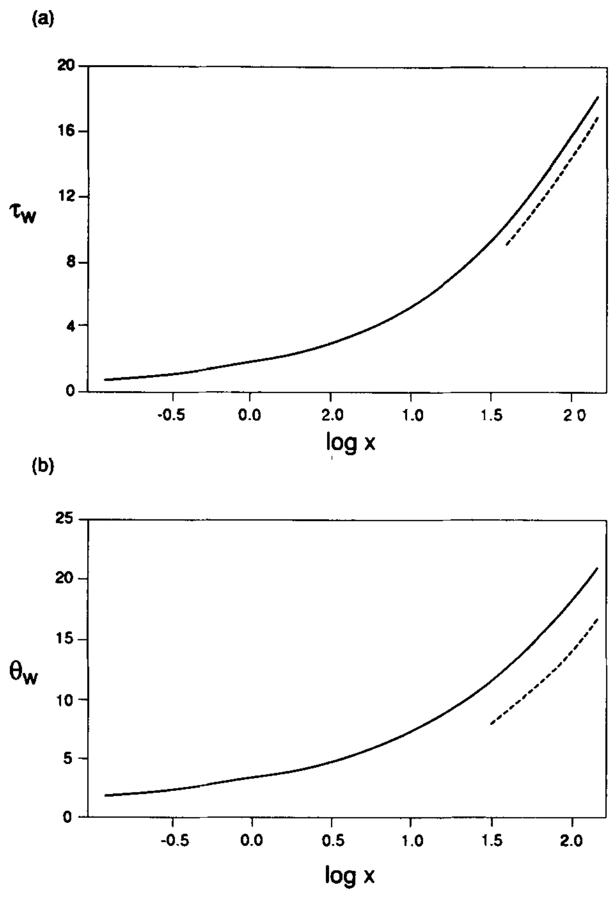

Fig. 10. Graphs of (a) skin friction τ_ω, (b) temperature θ_ω, obtained from the numerical solution of equations (4) and (58) with $\sigma = 1$. The asymptotic solution (79) is shown by the broken line.

To obtain a further relation between r and s we are guided by the previous work on the similarity solutions. Here we found that the perturbations to the leading order solutions arising from the wall heat flux boundary condition and from the viscous terms had to be of the same order. Applying this condition here gives $-s = s - r$, and hence $r = \frac{1}{2}$, $s = \frac{1}{4}$.

The above suggests that for this inner region we put

$$\psi = -x\phi(x, \zeta)\,, \quad \theta = x^{1/2} g(x, \zeta)\,, \quad \zeta = y/x^{1/4}\,, \tag{65}$$

with equations (4) becoming

$$f\frac{\partial^2 f}{\partial \zeta^2} - \frac{3}{4}\left(\frac{\partial f}{\partial \zeta}\right)^2 + g - x^{-1/4}\frac{\partial^3 f}{\partial \zeta^3} = x\left(\frac{\partial f}{\partial \zeta}\frac{\partial^2 f}{\partial \zeta\,\partial x} - \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial \zeta^2}\right), \tag{66a}$$

$$f\frac{\partial g}{\partial \zeta} - \frac{1}{2}g\frac{\partial f}{\partial \zeta} - \frac{x^{-1/4}}{\sigma}\frac{\partial^2 g}{\partial \zeta^2} = x\left(\frac{\partial f}{\partial \zeta}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial \zeta}\right), \tag{66b}$$

subject to the boundary conditions

$$f = 1\,, \quad \frac{\partial f}{\partial \zeta} = 0\,, \quad \frac{\partial h}{\partial \zeta} = -x^{-1/4} \quad \text{on} \quad \zeta = 0\,, \tag{67}$$

the outer boundary conditions are relaxed at this stage.

Equations (66, 67) suggest looking for a solution by expanding

$$f(x, \zeta) = f_0(\zeta) + x^{-1/4} f_1(\zeta) + \cdots\,,$$

$$g(x, \zeta) = g_0(\zeta) + x^{-1/4} g_1(\zeta) + \cdots\,. \tag{68}$$

At leading order we obtain the equations

$$f_0 g_0' - \frac{1}{2} f_0' g_0 = 0\,, \tag{69a}$$

$$f_0 f_0'' - \frac{3}{4} f_0'^2 + g_0 = 0\,, \tag{69b}$$

subject to

$$f_0(0) = 1\,, \quad f_0'(0) = 0\,, \quad g_0'(0) = 0\,, \tag{69c}$$

(primes now denote differentiation with respect to ζ). The solution of equation (69a) is

$$g_0 = \kappa f_0^{1/2}\,, \tag{70}$$

for some constant κ to be determined. Equation (69b) can then be solved to get

$$f_0' = -\sqrt{2\kappa}\,(f_0^{1/2} - f_0^{3/2})^{1/2}\,, \tag{71a}$$

and then implicitly to

$$\sqrt{2\kappa}\;\zeta = \int_{f_0}^{1} \frac{\mathrm{d}t}{(t^{1/2} - t^{3/2})^{1/2}}\,. \tag{71b}$$

Solution (71) satisfies $f_0(0) = 1$, $f_0'(0) = 0$ and has f_0 zero again at $\zeta = \zeta_0$ where

$$\zeta_0 = \frac{1}{\sqrt{2\kappa}} \int_0^1 \frac{dt}{(t^{1/2} - t^{3/2})^{1/2}} = \left(\frac{\pi}{2\kappa}\right)^{1/2} \frac{\left(-\frac{1}{4}\right)!}{\left(\frac{1}{4}\right)!}, \tag{72}$$

i.e. at $\zeta_0 = 1.695\kappa^{-1/2}$.

As for the similarity solutions the value of κ is determined from a consideration of the equations for next term in expansion (68). Again we need consider only the equation for g_1 which is, using (70),

$$f_0 g_1' - \frac{1}{4} f_0' g_1 = \frac{k}{2}\left(f_0^{1/2} f_1' - \frac{3}{4} f_0^{-1/2} f_0' f_1\right) + \frac{1}{\sigma} g_0''. \tag{73}$$

The solution of equation (73) can be expressed as

$$g_1 = A_1 f_0^{1/4} + \frac{\kappa}{2} f_0^{-1/2} f_1 - \frac{\kappa^2 f_0^{-1/4}}{4\sigma} \int_0^{\zeta} \frac{1 + f_0}{f_0^{9/4}} d\bar{\zeta}, \tag{74}$$

for some constant A_1. Now (74) satisfies the boundary condition $g_1'(0) = -1$ only if

$$\kappa = \sqrt{2\sigma}. \tag{75}$$

The solution given by (71) satisfies the outer boundary conditions only at a finite value ζ_0 of ζ, given by (72) and an outer shear layer is required to achieve a smooth transition to ambient conditions. To obtain this we need the behaviour of f_0 and g_0 as $\zeta \to \zeta_0$, which we find to be

$$f_0 \sim \left(\frac{9\sigma}{4}\right)^{2/3} (\zeta_0 - \zeta)^{4/3}, \quad g_0 \sim \sqrt{2\sigma}\left(\frac{9\sigma}{4}\right)^{1/3} (\zeta_0 - \zeta)^{2/3}. \tag{76}$$

We then require an outer (shear layer) region centred on $y = \zeta_0 x^{1/4}$ in which the viscous terms are important at leading order and which should match with (76). A consideration of the terms in equations (4) then suggests that we should put

$$\psi = x^{6/7} F(x, Y), \quad \theta = x^{3/7} G(x, Y), \quad Y = \bar{y}/x^{1/7}, \tag{77}$$

where $\bar{y} = y - \zeta_0 x^{1/4}$.

If we substitute (77) into equations (4) and use Prandtl's transposition theorem we arrive at equations for F and G which we solve by expanding in powers of $x^{-1/4}$. The leading order terms of this expansion, $F_0(Y)$ and $G_0(Y)$, satisfy the equations

$$F_0''' + G_0 + \frac{6}{7} F_0 F_0'' - \frac{5}{7} F_0'^2 = 0, \tag{78a}$$

$$\frac{1}{\sigma} G_0'' + \frac{6}{7} F_0 G_0' - \frac{3}{7} F_0' G_0 = 0, \tag{78b}$$

subject to the boundary conditions

$$F_0' \to 0, \quad G_0 \to 0 \text{ as } Y \to \infty, \tag{78c}$$

and, on matching with the inner region, that

$$F_0 \sim \left(\frac{9\sigma}{4}\right)^{2/3} (-Y)^{4/3}, \quad G_0 \sim \sqrt{2\sigma}\left(\frac{9\sigma}{4}\right)^{1/3} (-Y)^{2/3} \quad \text{as} \quad Y \to -\infty. \tag{78d}$$

The solution of the problem defined by equations (78) then completes the description of the asymptotic behaviour of the solution for blowing to leading order.

From (65), (70), (71) and (75) we have that

$$\tau_\omega \sim \sqrt{2\sigma} x^{1/2} + \cdots, \quad \theta_\omega \sim \sqrt{2\sigma} x^{1/2} + \cdots, \tag{79}$$

as $x \to \infty$. Graphs of τ_ω and θ_ω given by (79) are also shown in Fig. 10. We can see that the agreement between these asymptotic values and numerical solution is reasonable, better for τ_ω than for θ_w, though in both cases the convergence between the two sets of results is slow. This is to be expected as the error in (79) is $O(x^{1/4})$ and extremely large values of x, beyond the scope of the numerical integrations, are required before this is small in relation to terms of $O(x^{1/2})$.

5. Discussion

We have considered the effect that the injection or withdrawal of fluid has on free convection boundary layers on vertical surfaces with prescribed wall heat flux. We have examined two particular aspects of this problem in detail, namely those wall fluxes and transpiration velocities which give rise to a similarity transformation and the case of constant wall conditions.

In the former case we found that the solution depended on two basic parameters the power-law exponent n and a dimensionless transpiration parameter γ (as well as on the Prandtl number). We showed that the similarity equations had a solution (of the required forward boundary-layer type) only for $n > -1$ for all $\gamma > 0$ (blowing), this result being independent of the Prandtl number. For $\gamma < 0$ (suction) we found that a solution existed for all $n > n_0$, where n_0 was determined by the solution of a system of equations and depended on both γ and the Prandtl number, with $n_0 < -1$ for $\gamma < 0$. The range of existence of solution of the corresponding similarity equations for the prescribed wall temperature case, first proposed by Eichhorn [1], has not been examined. For the case without transpiration it has been shown in Merkin [12] that this range of existence depends on the Prandtl number and that the structure of the solution close to the corresponding n_0 is essentially different to that found for the prescribed heat flux case. Hence, it is reasonable to expect differences between the two cases when transpiration effects are included.

We then examined the behaviour for strong suction and for strong blowing. For strong suction, asymptotic profiles were found, similar in character to those found previously for the prescribed wall temperature case [3, 5]. However, for strong blowing the situation was essentially different to that found for the prescribed wall temperature case. The structure of the solution for large γ depended on the value of n, with different behaviour in the three ranges $-1 < n < -\frac{1}{4}$, $-\frac{1}{4} < n < \frac{7}{2}$, $n > \frac{7}{2}$. In the first case ($-1 < n < -\frac{1}{4}$) the boundary layer became very thin (with a thickness of $O(\gamma^{-1})$) and both the wall temperature and the skin friction became large, of $O(\gamma^4)$ and $O(\gamma^3)$ respectively. This feature has not been reported before for boundary layers subject to blowing through the wall where it is usually seen, for example in [6, 19, 20], that the fluid blown through the wall creates a thick inviscid inner region, with there then being a thinner outer (shear) layer before the ambient conditions are attained. The thinning of the boundary layer with n in this range for large injection rates is unexpected.

The normal situation of a thick inviscid inner region with a thin viscous shear layer seen previously for large injection rates is found to apply when n is in the range $-\frac{1}{4} < n < \frac{7}{2}$. However, for $n > \frac{7}{2}$ the boundary-layer structure is again different. There is still the inviscid inner region of thickness $O(\gamma^{1/4})$, but now the viscous effects are manifested in a thicker outer region, of extent of $O(\gamma^{1/4+(2n-7)/12})$. The solution for large injection rates has not been examined in detail for the prescribed wall temperature case and the question as to whether the form of the asymptotic structure is dependent on the value of n is still an open one.

We then considered the, perhaps, more realistic case of constant heat flux and constant transpiration velocity. This case does not admit a similarity solution and the governing equations have to be solved numerically. This was done using a standard numerical technique and presents no difficulties. We then derived solutions for large x (distance from the leading edge) for both suction and blowing. In the former case the solution followed fairly closely that found previously for the isothermal wall case [3]. However, differences between this and the present case were seen in the asymptotic solution for blowing. Here we found that the wall temperature became, large, of $O(x^{1/2})$, as $x \to \infty$. Also, we needed to be guided by our previous study of the similarity equations in order to ascertain correctly the structure of the asymptotic solution.

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